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On the meaning of the string-inspired noncommutativity and its implications

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Abstract: We propose an alternative interpretation for the meaning of noncommutativity of the string-inspired field theories and quantum mechanics. Arguments are presented to show that the noncommutativity generated in the stringy context should be assumed to be only between the particle coordinate observables, and not of the spacetime coordinates. Some implications of this fact for noncomutative field theories and quantum mechanics are discussed. In particular, a consistent interpretation is given for the wavefunction in quantum mechanics. An analysis of the noncommutative theories in the Schrödinger formulation is performed employing a generalized quantum Hamilton-Jacobi formalism. A formal structure for noncommutative quantum mechanics, richer than the one of noncommutative quantum field theory, comes out. Conditions for the classical and commutative limits of these theories have also been determined and applied in some examples.

Keywords: Topological Field Theories, Bosonic Strings, Non-Commutative Geometry.

Contents

1. Introduction

Recently, there has been a great interest in noncommutative theories of the canonical type. They are characterized by the following commutation relation

$$\left[\widehat{X}^{\mu}, \widehat{X}^{\nu}\right] = i\theta^{\mu\nu}\,, \tag{1.1}$$

where $\theta^{\mu\nu}$ is an antisymmetric constant tensor function of the coordinates.

During the latest years, a great deal of work and effort has been done in the direction of understanding this kind of noncommutativity and its implications in the context of quantum field theory and quantum mechanics (for good reviews see [1, 2]). Part of the interest in these theories is due to the fact that noncommutativity is present in certain models of string theory [3, 4], and M-theory [5, 6]. Another motivation comes from field theory itself. Semi-classical arguments, combining General Relativity and the Heisenberg uncertainty principle, also lead to noncommutative field theory [7]. The study of these theories gives us the opportunity to understand interesting phenomena, like nonlocality and IR/UV mixing [8], new physics at very short distances [2, 9], and possible implications of Lorentz violation [10].

But the interest in noncommutativity goes beyond formal manipulations. There are several lines of investigation on the possible phenomenology associated to the canonical noncommutativity, from cosmology and high energy scattering experiments to low energy precision tests [11].

The articles in this emerging branch of physics are growing up very fast. By carefully analyzing the literature in the area, it is possible to identify two primary research lines in canonical noncommutative quantum field theory. The first one presupposes a noncommutative spacetime from the beginning, being an intrinsic approach. It is closely related to the ideas of Connes noncommutative geometry [12]. This is the line followed, for example, by Chaichian et all [13], Cho et all [14] and Wulkenhaar [15].

The second research line has its roots in a low-energy approximation of string theory in the presence of a strong background B field [4]. This is the approach assumed, for example, in [8] and followers. It is also the one adopted in this work. Therefore, we will refer to the noncommutative quantum field theory (NCQFT) and the associated noncommutative quantum mechanics (NCQM) always adopting this point of view.

The implications of considering noncommutativity as property of the spacetime have been a subject under discussion. As it can be seen in [8], NCQFT is supported by the use of the Feynman graph approach for its calculations. It was argued in [15], for example, that such kind of procedure would not be expected from a theory defined on a true noncommutative spacetime, which not a manifold, at least in the usual sense. At this point, it is important to take into account that NCQFT comes from string theory, which is based on a continuous manifold structure. Therefore, it would be quite reasonable to interpret it as commutative as long as spacetime is concerned. This naturally leads to the formulation of a question: what kind of noncommutativity can be behind NCQFT, if not the one of spacetime?

The first goal of this work is to present an answer to the question above, proposing an alternative interpretation for the string-inspired noncommutativity and discussing its implications for NCQFT and NCQM. As we shall see later, one of these implications is the need of reinterpreting the meaning of the wavefunction. Since the interpretation for the noncommutativity proposed in this work is entirely new, we feel the necessity to analyze more carefully the NCQFT and NCQM formulations until now available in the literature. This is quite related to our second goal, which is to explore the Schrödinger formulation of NCQFT and NCQM, whose advantages to discuss some fundamental questions, like the quantum/classical and noncommutative/commutative passages, will be underlined in our exposition.

The organization of this work is the following. In section 2, we give a new interpretation for the meaning of the noncommutativity originated in the string theory context by tracing an intuitive parallel with the Landau problem. The Third section deals with NCQFT in the Schrödinger picture and the discussion of the non-triviality to achieve its commutative and classical limits. The remaining of the work is on NCQM. In the Fourth section, we show how to construct a consistent interpretation for NCQM in the Schrödinger formulation, a problem still now open in the literature. We also apply the method proposed in section 3 to carry out an analysis of the formal structure of quantum mechanics and to determine the conditions for the achievement of its classical and commutative limits. Two simple examples of application are given in section 5 to illustrate how the ideas proposed here work in concrete models. We end up in section 6 with a general discussion and outlook.

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2. The meaning of stringy noncommutativity

Let us discuss the origin of the noncommutativity relation (1.1) and its interpretation. For simplicity, we will restrict our work to the case where $\theta^{oi} = 0$. To grasp some intuition, we will appeal for the similarity between the derivation of the canonical noncommutativity in the stringy context and the one present in the Landau problem when a system is projected onto its lowest Landau level [1, 4]. Though being exhaustively explored in the literature, this analogy has always been centered in what is the origin of the noncommutativity, and not in what it is about.

We start by analyzing the Landau problem. Consider a nonrelativistic particle moving on the $x-y$ plane in the presence of a constant magnetic B field pointing in the z-direction. The classical lagrangian of the system is

$$L = \frac{1}{2}m\dot{\vec{x}}^2 + e\dot{\vec{x}} \cdot \vec{A}\,, \tag{2.1}$$

where e is the particle charge, m is the particle mass, and $\vec{A}$ is the electromagnetic vector potential. We are considering units where $c = 1$. The quantum hamiltonian is

$$\widehat{H} = \frac{1}{2m}\hat{\pi}^i\hat{\pi}^i\,, \tag{2.2}$$

where $\hat{\pi}^i = m\dot{\hat{x}}^i = \hat{p}^i - e\hat{A}^i$ are the physical momenta and $\hat{p}^i$ are the canonical momenta. Notice that the canonical momenta commute, while the physical momenta satisfy the commutation relation

$$\left[\hat{\pi}^i, \hat{\pi}^j\right] = i\hbar eB\epsilon^{ij}\,. \tag{2.3}$$

To understand how the noncommutativity (1.1) arises in the model it is useful to define, in analogy with the classical case, the center-of-orbit operator, whose components are given by

$$\widehat{X}^i = \hat{x}^i - \frac{i}{eB}\hat{\pi}^i\,. \tag{2.4}$$

These components can be shown to satisfy the commutation relation

$$\left[\widehat{X}^i, \widehat{X}^j\right] = -i\frac{\hbar\epsilon^{ij}}{eB} = i\theta^{ij}\,, \tag{2.5}$$

where $\theta^{ij} = (-\hbar/eB)\,\epsilon^{ij}$. Notice that while $[\hat{x}^i, \hat{x}^j] = 0$, the $\widehat{X}^i$ are not allowed to commute due to the presence of the term containing the magnetic field. The spacetime, on the other hand, is the same ordinary commutative one. In this scenario, the uncertainty relation

$$\Delta X^i \Delta X^j \geq \frac{\hbar}{2}\left|\frac{\epsilon^{ij}}{eB}\right|\,, \tag{2.6}$$

introduced by (2.5), must be understood as just being a consequence of the limitation on the information available about the X^i coordinates in a process of measurement.

Now we consider the strong magnetic field limit. In this case, the system is projected onto the lowest Landau level. A rigorous prescription of how to work in this limit, which is achieved by solving the constraint $\hat{\pi}^i \approx 0$ (using a projection technique), may be found in [16]. On heuristic grounds, one can understand the projection onto the lowest Landau level as a process where the particles have their kinetic degrees of freedom frozen and are confined into their respective orbit centers [17]. The particle coordinate observables in this limit clearly satisfy (2.5) as a consequence of the coincidence between $\widehat{X}^i$ and $\hat{x}^i$, but there is no fundamental reason to believe that the spacetime coordinates will become noncommutative in this limit as a consequence of the projection. In reality the spacetime should be assumed as being unaltered, instead of becoming fuzzy or pointless.

Usually the relation (2.5) is achieved in the literature by dropping the kinetic term directly from the lagrangian (2.1). If we write the vector potential as $\overrightarrow{A} = (0, Bx, 0)$, and consider the $B \to \infty$ or $m \to 0$ limits, we can discard the kinetic term and write the lagrangian as

$$L = eBx\dot{y} \,. \tag{2.7}$$

In this lagrangian, the x and y variables are canonically conjugate, and their respective quantum operators satisfy a commutation relation identical to (2.5),

$$\left[\hat{x}^i, \hat{x}^j\right] = -\frac{i\hbar}{eB}\epsilon^{ij} \,. \tag{2.8}$$

Notice that, when the discussion is directed from this point of view, the derivation of the canonical noncommutativity deviates attention from a fundamental point: the noncommutativity obtained as a consequence of projection onto the lowest Landau level (which enforces the identification between $\widehat{X}^i$ and $\hat{x}^i$) ought to be in reality assumed as a property of the particle coordinate observables. This is the source of some confusion in the literature on the Landau problem, and also in the high-energy-physics articles on NCQFT that refer to it. For instance, take for example the following assertion found in [18]: "An example of a system where space-time coordinates do not commute is that of a particle in a strong magnetic field". One could ask how a strong magnetic field in a laboratory could turn the spacetime noncommutative and thus destroy its pointwise structure.

The canonical noncommutativity originated from string theory in [4] is based on an approximation which similar to the one of the lowest Landau level just described. Consider open bosonic strings moving in a flat euclidean space with metric $g_{\mu\nu}$, in the presence of a constant Neveu-Schwarz B-field and with Dp branes. The B-field probed by the open strings is equivalent to a constant magnetic field on the branes, and it can be gauged away in the directions transverse to the Dp brane worldvolume. The worldsheet action is

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'}\int_\Sigma \left(g_{\mu\nu}\partial_\eta X^\mu \partial^\eta X^\nu - 2\pi i\alpha' B_{\mu\nu}\epsilon^{\eta\lambda}\partial_\eta X^\mu \partial_\lambda X^\nu\right) \\ &= \frac{1}{4\pi\alpha'}\int_\Sigma g_{\mu\nu}\partial_\eta X^\mu \partial^\eta X^\nu - \frac{i}{2}\int_{\partial\Sigma} B_{\mu\nu}X^\mu \partial_t X^\nu \,, \end{aligned} \tag{2.9}$$

where $\alpha' = l_s^2$, Σ is the string worldsheet, ∂_t is a tangential derivative along the worldsheet boundary $\partial\Sigma$ and the X^μ is the embedding function of the strings into flat spacetime.

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If we consider the limit $g_{\mu\nu} \sim (\alpha')^2 \to 0$, keeping $B_{\mu\nu}$ fixed [4], the bulk kinetic terms of (2.9) vanish. The worldsheet theory in this limit is topological. All that remains are the boundary degrees of freedom, which are governed by the action

$$S = -\frac{i}{2} \int_{\partial\Sigma} B_{\mu\nu} X^{\mu} \partial_t X^{\nu} \,. \tag{2.10}$$

If one regards (2.10) as a one dimensional action, and ignores the fact that the $X^{\mu}(t)$ are the endpoints of a string, then it can be considered as analogous to the action corresponding to the lagrangian of the Landau problem (2.7). Under the approximation being considered, the $X^{\mu}(t)$ may be regarded as operators satisfying the canonical commutation relation

$$\left[\widehat{X}^{\mu}, \widehat{X}^{\nu}\right] = \left(\frac{i}{B}\right)^{\mu\nu}, \tag{2.11}$$

which is identical to the one of (1.1) by defining $\theta^{\mu\nu} = (1/B)^{\mu\nu}$.

The close similarity between this heuristic derivation[1] and the one presented for the Landau problem suggests that the noncommutativity achieved by the approximation under consideration may be assumed to be of the particle coordinate observables, instead of the spacetime coordinates. At this point, it is interesting to quote that other noncommutative theories which emerged in the stringy context, known as noncommutative dipole theories [19], were presented that contain a noncommutativity also not of the spacetime coordinates. In these theories noncommutativity is an inherent property of the fields.

Although the great majority of papers on NCQFT consider the spacetime as being "pointless", they use the basis of plane wave for their calculations in momentum space (see for example [1, 2, 4, 8] and references therein) and interpret the space of Weyl symbols [13] as the physical position space. This is equivalent to allow the localization of information at individual points in the spacetime. The apparent contradiction in this procedure is clearly shown to be absent if we interpret the stringy spacetime that they are considering as commutative. It is important to keep in mind that the distinction between the intrinsic spacetime noncommutativity and the coordinate observable noncommutativity is not a question of metaphysics. Since the calculation of the Green functions in the first case must employ the procedure of averaging over localized states in the space of Weyl symbols [13] (which smears the pointwise information), the finiteness of the theories, as well as the measurable physical quantities, will differ according to the approach adopted for the noncommutativity.

Let us review the Weyl quantization procedure, considering stringy noncommutativity from the new point of view. For the case under consideration, it consists in the establishment of a map from the noncommutative space of the particle coordinate observables to the space

[1] We abbreviated the exposition to go directly to the point of interest of this work: the analogy of the stringy noncommutativity to the one of the Landau problem. This is an issue previously pointed out, for example, in [1, 4]. A detailed discussion on the derivation of (1.1) would involve certain subtleties, and may be found in those references.

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of commutative coordinates by the use of the Moyal star product

$$
\begin{aligned}
(f \star g)(x) &= \frac{1}{(2\pi)^n} \int d^n k d^n p e^{i(k_\mu + p_\mu)x^\mu - \frac{i}{2} k_\mu p_\nu \theta^{\mu\nu}} f(k) g(p) \\
&= e^{\frac{i}{2}\theta^{\mu\nu} \frac{\partial}{\partial \xi^\mu} \frac{\partial}{\partial \eta^\nu}} f(x+\xi) g(x+\eta) \Big|_{\xi=\eta=0} .
\end{aligned} \tag{2.12}
$$

According to this technique, the representation of a given NCQFT originally valued on the space of noncommutative variables on the space of ordinary commutative functions is achieved by the replacement of the noncommuting variables in the arguments of the fields in the action by the commutative ones and the ordinary product of the fields by the Moyal product. Since the noncommutativity is assumed to be only of the particle coordinate observables, what the Weyl procedure does is just to represent the field theory originally valued on this noncommutative variables in function of the spacetime coordinates and canonical momenta. This will become particularly clear later, when we will consider NCQM.

Due to the property $\int d^4x A \star B = \int d^4x\ AB$, the Hilbert (Fock) space of a noncommutative field theory can be chosen to be the same as the one of its commutative counterpart at the perturbative level [20], and the noncommutativity is manifest only through the interaction terms of the action. This gives us no surprise, since the difference of a commutative quantum field theory and its noncommutative counterpart is only the presence of the strong Neveu-Schwartz B-field in the background, acting to forbid the simultaneous measurement of all the position coordinates of the particles. Of course, the $B-$ field does this by interacting with the quantum fields, and thus its presence must be encoded in the interaction terms of the action.

3. NCQFT and the Hamilton-Jacobi theory

Here we discuss some aspects of the Schrödinger representation for NCQFT based on the canonical deformation of the algebra of the coordinate observables. A study of NCQFT in the Heisenberg and path integral formulations may be found in [21].

For simplicity, we will consider scalar field theories with action

$$
S = \int d^4x \left\{ \frac{1}{2} \left(\partial^\mu \varphi \partial_\mu \varphi - \frac{m^2}{\hbar^2} \varphi^2 \right) - V_\star(\varphi) \right\}, \tag{3.1}
$$

where $V_\star(\varphi)$ is the usual commutative interaction part of the potential with the star product replacing the ordinary one. Since the discussion here is general, and the procedure presented can be applied to dimensions other than four, we will keep the potential as a generic real analytic one. Of course, the question of renormalizability must be taken into account when choosing the acceptable potentials [22]. All the quantization procedures of the usual formalism developed for the commutative quantum field theory in the Schrödinger representation [23] are similar in the noncommutative case under consideration, which presupposes that $\theta^{0i} = 0$. We briefly present the aspects of the procedure that will be useful in this work.

The hamiltonian corresponding to (3.1) is

$$H=\int d^3x\left\{\frac{1}{2}\left(\pi^2+|\nabla\varphi|^2+\frac{m^2}{\hbar^2}\varphi^2\right)+V_\star(\varphi)\right\}, \tag{3.2}$$

where $\pi(x)=\dot{\varphi}(x)$ is the conjugate field momentum. The operators $\widehat{\varphi}(x)$ and $\widehat{\pi}(x)$ satisfy the canonical equal-time commutators,

$$\begin{aligned}\left[\widehat{\varphi}(\vec{x},t),\widehat{\pi}(\vec{y},t)\right]&=i\hbar\delta(\vec{x}-\vec{y}),\\ \left[\widehat{\varphi}(\vec{x},t),\widehat{\varphi}(\vec{y},t)\right]&=\left[\widehat{\pi}(\vec{x},t),\widehat{\pi}(\vec{y},t)\right]=0.\end{aligned} \tag{3.3}$$

To work in a coordinate field representation, we shall consider a basis for the Fock space where the operator $\widehat{\varphi}$ is time independent and diagonal $\widehat{\varphi}(\vec{x})\mid\phi\rangle=\phi(\vec{x})\mid\phi\rangle$. In this basis, the state $\mid\Psi\rangle$ is represented by the time dependent wave functional $\Psi[\phi,t]=\langle\phi\mid\Psi\rangle$, and the momentum field operator $\hat{\pi}(\vec{x})$ by $-i\hbar\delta/\delta\phi(\vec{x})$. The Schrödinger equation is written as

$$i\hbar\frac{\partial}{\partial t}\Psi[\phi,t]=\int d^3x\left\{\frac{1}{2}\left(-\hbar^2\frac{\delta^2}{\delta\phi(\vec{x})^2}+|\nabla\phi|^2+\frac{m^2}{\hbar^2}\phi^2\right)+V_\star(\phi)\right\}\Psi[\phi,t]. \tag{3.4}$$

If we write the wavefunctional in its polar form $\Psi=R\exp(iS/\hbar)$ and substitute it into (3.4), we obtain

$$\begin{aligned}\left[i\hbar\frac{\partial R}{\partial t}-R\frac{\partial S}{\partial t}\right]e^{iS/\hbar}=&\int d^3x\left\{-\frac{\hbar^2}{2}\left[\frac{\delta^2R}{\delta\phi^2}-\frac{R}{\hbar^2}\left(\frac{\delta S}{\delta\phi}\right)^2+i\left(\frac{2}{\hbar}\frac{\delta R}{\delta\phi}\frac{\delta S}{\delta\phi}+\frac{R}{\hbar}\frac{\delta^2S}{\delta\phi^2}\right)\right]\right\}e^{\frac{i}{\hbar}S}+\\ &+\int d^3x\left[\frac{1}{2}|\nabla\phi|^2+\frac{m^2}{2\hbar^2}\phi^2+V_\star(\phi)\right]Re^{iS/\hbar}.\end{aligned} \tag{3.5}$$

Dividing (3.5) by Ψ and separating the real part, we find

$$\frac{\partial S}{\partial t}+\int d^3x\left\{\frac{1}{2}\left[\left(\frac{\delta S}{\delta\phi}\right)^2+|\nabla\phi|^2+\frac{m^2}{\hbar^2}\phi^2\right]+V(\phi)\right\}+V_{nc}[\phi]+Q_K[\phi]=0, \tag{3.6}$$

where

$$V_{nc}[\phi]=\int d^3x\left\{\mathrm{Re}\left[V_\star(\phi)\right]-V(\phi)\right\}, \tag{3.7}$$

and

$$Q_K[\phi]=-\frac{\hbar^2}{2R}\int d^3x\frac{\delta^2R}{\delta\phi(\vec{x})^2}. \tag{3.8}$$

Equation (3.6) is the ordinary Hamilton-Jacobi equation for the scalar field with two additional potentials V_{nc} and Q_K.

The complete information contained in the functional Schrödinger equation is extracted only if we consider (3.6) and the other coupled equation that is obtained from the decomposition of (3.4) into its real and imaginary parts. Multiplying (3.5) by $2R/\hbar\exp(-iS/\hbar)$, taking the imaginary part, and doing some simplification using the properties of the star product, the equation obtained reads

$$\frac{\partial R^2}{\partial t}+\int d^3x\frac{\delta}{\delta\phi}\left(R^2\frac{\delta S}{\delta\phi}\right)=0. \tag{3.9}$$

This expression must be interpreted as a continuity equation for the probability density $R^2[\phi(\vec{x}), t]$ that the field configuration be $\phi(\vec{x})$ at time t. Notice that (3.6) and (3.9) constitute a set of nonlinear coupled equations. In practice, it is convenient to solve directly (3.4) and then obtain R and S from the wavefunctional Ψ. The advantage of decomposing the functional Schrödinger equation in a more complex system of two coupled equations is for the sake of its physical interpretation.

In what follows, we shall use the generalized Hamilton-Jacobi formalism introduced to perform an analysis of the conditions for a system achieve its classical and commutative limits. Before going on, however, it is necessary to define precisely what we mean for achieving these limits.

Finding the classical or commutative limit of a system is to establish the conditions that must be imposed on the environment, quantum numbers or physical constants in order to enforce the system to assume the behavior identical to the one of a classical or commutative analog. These conditions can be determined by analyzing equations (3.6) and (3.9). When we consider a solution to the equations (3.6) and (3.9), and imagine it as being substituted on them, the original set of equations must now be understood as a set of identities that are being trivially satisfied. The classical or commutative limits are achieved when the conditions are such that this set of equations (seen as a set of identities) assume the classical form or the quantum commutative form. Notice that, according to this criterion, the establishment of the adequate prescription for the achievement of the limits of each individual system must be done by accounting for the properties that characterize its physical state. The procedure proposed will thus lead to the determination of state dependent criteria, thought they can be written in an universal and compact form, as will be shown later.

The first interesting fact to be noticed once one adopts the prescription suggested here, is that the achievement of the classical limit will, in certain cases, occur under conditions that differ form the naive one, $\hbar \to 0$, usually employed in many textbooks. Here, and in what follows, the symbol "$A \to B$" must be understood as taking a limit such that $A - B$ is sufficiently small to be neglected in comparison with the other quantities which are relevant to the physical system under consideration. For example, $\hbar \to 0$ may be assumed to mean that $\hbar \ll S_0$, where S_0 is some characteristic action of the corresponding classical motion of the system [24, 25]. In the achievement of the classical limit starting from (3.6) and (3.9), we must take into account the fact that R (and S) in principle depends on $\hbar$, and hence in general the condition $Q_K \to 0$ is not satisfied always when $\hbar \to 0$. A detailed discussion about the fallacy in adopting $\hbar \to 0$ as the universal criterion for the classical limit in the context of ordinary quantum mechanics and field theory containing examples is found in [26, 27] and references therein.

An analogous reasoning can be applied when we consider the commutative limit. Although $V_{nc} \to 0$ when $\theta^{ij} \to 0$, as can be seen from (2.12) and (3.7), this criterion may not be valid for the achievement of the commutative limit. This is a consequence of the fact that Q_K may contain contributions generated by the noncommutativity that do not depend on θ^{ij}. On that case, taking $\theta^{ij} \to 0$ will not vanish them, do not conducting to the commutative limit. Moreover, since in principle the dependence of Q_K on θ^{ij} is arbitrary and totally state dependent, the possibility of Q_K blowing-up when $\theta^{ij} \to 0$ must not be discarded. In reality this is expected to occur in some models.

When we consider a differential equation which is deformed depending on a set of parameters (for example θ^{ij}), the set of admissible solutions is richer than the original one corresponding to the undeformed equation. If we smoothly undo the deformation, part of the expanded set of solutions will of course suffer a process of homotopy, and in the end of it will be contained in the set of solutions of the original undeformed equation. But part of the same set will not present the same behavior, and may become unaltered, go to another limit outside the set of undeformed solutions, or blow-up.

In fact, it has been verified that in many of the perturbative loopwise calculations of NCQFT, some peculiar phenomena occurs when $\theta^{ij} \to 0$, like UV/IR mixing [8], which is responsible for the non-analytic behavior in the $\theta^{ij} \to 0$ limit, the blowing-up of the self-energy in some models [28], non-reduction of some gauge theories to the commutative counterparts [29], among others [2]. In the quantum-mechanical context, it is interesting to quote the blowing-up of the energy spectra [30] and the appearance of wavefunctions representing extremely localized states [9, 31]. These states are particularly interesting as they present nonperturbative effects in θ^{ij} at small distances. In addition, if one starts considering these (normalizable) wavefunctions and consider the limit where θ^{ij} vanishes, they become Dirac delta functions, which are non-normalizable [31].

In what follows, we will turn operational the technique proposed for the determination of the classical and commutative limits by showing how it can be applied to perform an analysis of NCQFT. The origin of the quantum contributions on (3.6) is traced back to the action of the field momentum in its differential representation, $\hat{\pi}(\vec{x}) = -i\hbar\delta/\delta\phi(\vec{x})$, on the wavefunctional $\Psi = R\exp(iS/\hbar)$ in (3.4). When this equation is divided by Ψ and the real part is taken, the quantum Hamilton-Jacobi equation (3.6) is originated containing terms that cannot appear on the classical Hamilton-Jacobi equation, where the field momentum is represented by $\pi(\vec{x}) = \delta S/\delta\phi(\vec{x})$. These terms are nonclassical, and thus must be responsible by the quantum effects. This is a simple and safe criterion for the identification of the quantum contributions.

In practice, it is not necessary to follow all the steps and write the Hamilton-Jacobi equation containing all its terms. When one wants to identify and compute the quantum effects corresponding to an specific part of the Schrödinger hamiltonian, the procedure can be applied directly to the term under consideration to extract its quantum contributions, which are grouped to constitute its associated quantum potential. In equation (3.4), for example, the only term that has derivatives (here functional) acting on Ψ is kinetic one. Thus, there will be only one quantum potential, which must be associated with this term and is given by

$$\begin{aligned} Q_K &= \mathrm{Re}\left(-\frac{\hbar^2}{2}\frac{1}{\Psi}\int d^3x \frac{\delta^2\Psi}{\delta\phi^2}\right) - \left(\frac{\hbar^2}{2}\int d^3x \left(\frac{\delta S}{\delta\phi}\right)^2\right) \\ &= -\frac{\hbar^2}{2R}\int d^3x \frac{\delta^2 R}{\delta\phi(\vec{x})^2}\,. \end{aligned} \tag{3.10}$$

This reproduces (3.8), and is the expression used to define the quantum potential in the de Broglie-Bohm interpretation of quantum theory[2] [27].

It is possible to associate a quantum potential to each term or, according to the necessity, a group of terms involving derivatives in the hamiltonian. To suppress its corresponding quantum effects, it is sufficient to impose the cancelation of its corresponding quantum potential as an equation. The classical limit of the theory is found when the sum of all the quantum potentials vanish. In this case the Hamilton-Jacobi equation looses its dependence on R and on second and higher order derivatives of S, decouples from (3.9) and assumes its classical form, which is an equation exclusively for S. This is a general result, totally independent of the specific form of equation (3.6). By definition, the classical contributions are the ones obtained by the replacement $\pi(\vec{x}) = \delta S/\delta\phi(\vec{x})$, and thus do not contain any derivative of R that could spoil the decoupling process of the Hamilton-Jacobi equation. Once (3.6) and (3.9) assume a form characteristic of the classical physics, the physical content of the system is the same of the classical field theory, and therefore the observable quantities, as well as the equations of evolution of their averages, will be indistinguishable of the ones of classical physics.

Having determined the criterion for the classical limit as $Q_K \to 0$, let us concentrate our attention on the criterion for the commutative one. Since equation (3.9) is still in the ordinary commutative classical form, the discussion about how to achieve the commutative limit reduces to the analysis of the potentials Q_K and V_{nc} of (3.6). One safe way for the identification of the noncommutative effects present in Q_K is to calculate the same quantum potential with respect to the analog wavefunctional corresponding to the associated commutative field theory and compare with it. This is done by defining the commutative quantum potential as

$$Q_c = -\frac{\hbar^2}{2R}\int d^3x \frac{\delta^2 R_c}{\delta\phi(\vec{x})^2}\,, \tag{3.11}$$

where $R_c = \sqrt{\Psi_c^*\Psi_c}$ and Ψ_c is the solution of (3.4) with $\theta^{ij} = 0$, that is , the solution of the commutative counterpart. The noncommutative contributions contained in Q_K can thus be attributed to the functional

$$Q_{nc} = Q_K - Q_c\,, \tag{3.12}$$

which we define as the noncommutative quantum potential.

When $V_{nc} + Q_{nc} \to 0$, (3.6) becomes similar to the Hamilton-Jacobi equation corresponding to the conventional quantum field theory, assuring the noncommutative/commutative passage. This can happen in two ways:

[2]Usually, the polar decomposition of the wavefunctional and the associated Hamilton-Jacobi formalism are explored by the Bohmian community in the direction of establishing an ontological meaning for quantum field theory, with an additional assumption about the objective existence of quantum fields independent of the act of observation [26, 27]. This assumption, however, is not being considered in this work. The most general case where we intend that this method could be applied may not yet admit an ontological interpretation.

1. $V_{nc}+Q_{nc} \to 0$ is achieved by taking the limit $\theta^{ij} \to 0$ directly without any obstruction caused by the quantum effects represented by Q_{nc}. In this case the θ^{ij} disappear completely from all terms of the Hamilton-Jacobi equation. This is the most common approach and easy to understand in comparison with classical noncommutative field theory, where Q_{nc} is absent the commutative limit is usually achieved by taking $\theta^{ij} \to 0$.

2. $V_{nc}+Q_{nc} \to 0$ is achieved by varying a physical constant or parameter of the system, keeping θ^{ij} unaltered. This approach is less common, but may be useful specially in cases where the first is not possible to be realized.

Two examples of application of the Hamilton-Jacobi formalism in the quantum mechanical context are presented an worked out in the end of the article.

4. Noncommutative quantum mechanics

Here, we discuss the implications of considering the coordinate observables of the particles as operators satisfying (1.1) for quantum mechanics. We shall show how to work in the Schrödinger formulation with a consistent interpretation for the wavefunction and how the Hamilton-Jacobi formalism of the previous section can be applied to give information about the formal structure of NCQM in comparison with the one of NCQFT.

4.1 Establishing the foundations

As in the commutative theories, NCQM can be derived from NCQFT in its the low energy limit. A rigorous treatment for this derivation was done P. M. Ho and H. C. Kao in [32]. For chargeless particles their noncommutative Schrödinger equation reduces to

$$i\hbar \frac{\partial \Psi\left(x^i,t\right)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi\left(x^i,t\right) + V\left(x^i\right) \star \Psi\left(x^i,t\right). \tag{4.1}$$

By using the properties of the star product, the potential term of the Schrödinger equation can also be written as

$$V(x^i) \star \Psi\left(x^i,t\right) = V\left(x^i + \frac{i}{2}\theta^{ij}\partial_j\right)\Psi\left(x^i,t\right). \tag{4.2}$$

The hamiltonian is then

$$H = \frac{-\hbar^2}{2m}\nabla^2 + V\left(x^i + \frac{i}{2}\theta^{ij}\partial_j\right). \tag{4.3}$$

Thus, the noncommutative quantum mechanics is physically equivalent to a commutative quantum mechanics with new momentum dependent interactions, dictated by the replacement $x^i \to x^i - \theta^{ij}\hat{p}_j/2\hbar$ in the potential term. This can be assumed to be the Weyl correspondence for quantum mechanics, and has been widely used in the literature (see, for example [33]). However, the complete investigation of all of its consequences has not ever been done.

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To understand physics that lies behind the correspondence above, we again appeal to the Landau problem, this time do not considering the projection onto the lowest Landau level. An analogy between the physical variables suggests that the shift $x^i \to x^i - \theta^{ij}\hat{p}_j/2\hbar$ caused by the external Neveu-Schwartz B field is similar to the shift $\hat{p}^i \to \hat{p}^i - e\hat{A}^i$, which is caused by an external magnetic field in ordinary quantum mechanics. Since in the Landau problem the physical momentum is $\hat{\pi}^i = \hat{p}^i - e\hat{A}^i$, according to the analogy, the physical position must be[3] $\widehat{X}^i = x^i - \theta^{ij}\hat{p}_j/2\hbar$. This intuitively explains why the position observables must remain noncommuting after the Weyl correspondence is applied. This map is nothing but a representation of the noncommuting position observables in terms of the canonical momentum $\hat{p}^i$ and canonical coordinates x^i, the latter having the meaning of being points of the real physical space.

Some physical intuition on the meaning of the canonical position variables can be obtained by considering the dipole picture [35]. For the case of the NCQM under consideration, this consists in considering that, instead of a particle, the elementary object of the theory is a half dipole[4] whose extent is proportional to its momentum, $\Delta x^i = \theta^{ij}p_j/2\hbar$. One of the endpoints of the dipole carries its mass and is responsible for its interactions. The other extreme is empty. According to this intuitive view, the change of variables $X^i = x^i - \theta^{ij}p_j/2\hbar$ corresponds to a change of coordinates of the interacting extreme of the dipole X^i, where the corresponding physical particle is located, to its empty one x^i. Therefore, the effect of the background field, which acts to forbid the coordinates to commute, is compensated in this new commutative coordinate system obtained by change of variable that is dependent on the momentum and background intensity (remember $\theta^{ij} = (1/B)^{ij}$). Although the simultaneous measurement of the observables $\hat{x}^i$ is in principle possible, it is not sufficient for the determination of the physical position of the associated particle. As $\hat{x}^i$ and $\hat{p}^i$ do not commute, the knowledge of the three x^i forbids the simultaneous determination of the three p^i and thus the physical location of the particle.

The Hilbert space of states of noncommutative quantum mechanics is assumed to be the same of the commutative one [33], which is a remnant from a result of field theory discussed in the Second section. By knowing the Hilbert space of states and the Schrödinger evolution equation for the wavefunction, it remains to show that this wavefunction admits the definition of a density of probability. The usual definition of probability density

$$\rho = \Psi^*\Psi = |\Psi|^2 \tag{4.4}$$

can be employed with some care in interpreting its meaning. The wavefunction here is valued on the canonical coordinates x^i, rather than on eigenvalues of the physical position observables $\widehat{X}^i$. Thus $\rho(\vec{x},t)d^3x$ must be interpreted as the probability of finding the canonical coordinate of the particle in the volume d^3x around the point $\vec{x}$ at time t.

[3]The association of the $\widehat{X}^i$ with the observables corresponding to the physical coordinates of the particles was also proposed in [34], but following the same line of [33], which interpret the canonical positions x^i just as auxiliary variables, do not having any physical interpretation.

[4]In the context of this work, which considers point particles as the true elementary constituents of matter, the dipoles are assumed to be fictitious.

To extract information on the physical coordinate position of the particles from the wavefunction, it is necessary to expand it in eigenfunctions of the position operators $\widehat{X}^i = x^i + i\theta^{ij}\partial_j/2$. Of course, since the $\widehat{X}^i$ do not commute, it is impossible to find a basis for all the spatial directions simultaneously. This is the simple manifestation of the fact that, although the spacetime is endowed with a pointwise structure, it is impossible to localize a particle on a given point of it in a measurement. The background Neveu-Schwartz field always acts conspiring against any attempt in this direction, enforcing the results of the measurements to obey the uncertainty relation

$$\Delta X^i \Delta X^j \geq \frac{\left|\theta^{ij}\right|}{2}\,. \tag{4.5}$$

This solves the open question about the possibility of the existence of the wavefunction and how to interpret it [33, 36] in the context of the string- inspired noncommutativity. Had we considered the wrong interpretation of the canonical position observables $\hat{x}^i$ as the physical ones, the uncertainty relation (4.5) would be lost in the Schrödinger formulation. Of course, due to the smallness expected for θ^{ij}, in practice the noncommutative effects are not possible to be perceived by a direct verification of (4.5), which would demand an energy to work on a length scale far beyond the limits of validity of nonrelativistic quantum mechanics, but through its indirect consequences for other observable quantities, like atomic energy spectrum [33, 37], phase shifts on the Aharonov -Bohm effect [38], etc.

By differentiating (4.4) and using (4.1) we obtain

$$\frac{\partial\rho}{\partial t} = \frac{i\hbar}{2m}\nabla\cdot\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) - \frac{i}{\hbar}\left(\Psi^* V\star\Psi - \Psi\left(V\star\Psi\right)^*\right), \tag{4.6}$$

where the term $i\hbar\left(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi\right)/2m$ is well known from ordinary quantum mechanics and corresponds in that case to the current density $\vec{J}$. Notice that, since the canonical coordinates x^i do not represent the physical coordinates of the particles, the fact that the usual continuity equation $\partial\rho/\partial t + \nabla\cdot\vec{J} = 0$ is not satisfied does not mean that there is no local conservation law of probability. Such form of continuity equation would be expected only if $\left|\Psi(x^i,t)\right|^2 d^3x$ had the meaning of probability of finding the particle in a volume d^3x around the point $\vec{x}$ at time t.

Actually, the extra term that appeared in (4.6) due to the noncommutativity is not responsible for any inconsistency of the theory, but its role must be understood. One way to justify its presence is to notice that it is essential in order to assure that the *equivariance* property [24] is satisfied by $\rho = |\Psi|^2$. This means that if $\rho(x^i,t_0) = \left|\Psi(x^i,t_0)\right|^2$ at some time t_0, then $\rho(x^i,t) = \left|\Psi(x^i,t)\right|^2$ for all t. In other words, $\rho(x^i,t)$ preserves its form as a functional of $\Psi(x^i,t)$ for all times. This property is trivially satisfied because the starting point for the derivation of the equation (4.6) was exactly the definition of $\rho(x^i,t) = \left|\Psi(x^i,t)\right|^2$ as the density of probability for an arbitrary time and the noncommutative Schrödinger equation. Thus it is the noncommutative Schrödinger equation itself that enforces the appearance of the noncommutative term in the local probability conservation law for a question of consistency. This term could called "*equivariance* correction", and be denoted by Σ_θ. It should be expected, however, that when integrated over all the space Σ_θ

vanish, as occurs with the term containing the divergence. This is a necessary condition for the global conservation of probability. By integrating equation (4.6) over the space and using the properties of the star product it is easy to verify that

$$\int \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \left[\frac{i\hbar}{2m} \left(\Psi \nabla \Psi^* - \Psi^* \nabla \Psi \right) \right] \right) d^3x = -\frac{i}{\hbar} \int \left[\Psi^* V \star \Psi - \Psi \left(V \star \Psi \right)^* \right] d^3x = 0 \,, \tag{4.7}$$

thanks to the antisymmetry of $\theta^{\mu\nu}$.

4.2 Analyzing from Hamilton-Jacobi point of view

The Hamilton-Jacobi formalism associated with the NCQM is found by applying the same procedure previously discussed when considering the NCQFT. We write the wavefunction in its polar form $\Psi = R\, e^{iS/\hbar}$, substitute into equation (4.1), and separate its real and imaginary parts. For the real part we obtain

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V + V_{nc} + Q_K + Q_I = 0 \,. \tag{4.8}$$

The three new potential terms are defined as

$$V_{nc} = V \left(x^i - \frac{\theta^{ij} \partial_j S}{2\hbar} \right) - V \left(x^i \right) , \tag{4.9}$$

$$Q_K = \mathrm{Re} \left(-\frac{\hbar^2}{2m} \frac{\nabla^2 \Psi}{\Psi} \right) - \left(\frac{\hbar^2}{2m} \left(\nabla S \right)^2 \right) = -\frac{\hbar^2}{2m} \frac{\nabla^2 R}{R} \,, \tag{4.10}$$

and

$$Q_I = \mathrm{Re} \left(\frac{V \left(x^i + \frac{i}{2} \theta^{ij} \partial_j \right) \Psi}{\Psi} \right) - V \left(x^i - \frac{\theta^{ij}}{2\hbar} \partial_j S \right) . \tag{4.11}$$

V_{nc} is the potential that accounts for the noncommutative classical interactions, while Q_K and Q_I for the quantum effects. The origin of the Q_K potential comes from the kinetic term of the hamiltonian, closely related to the case of NCQFT discussed before. Its definition is identical to the one of the quantum potential in ordinary Bohmian mechanics [26], and here we will call it the kinetic quantum potential. The remaining term, Q_I, is the potential that accounts for the quantum effects that come from the interaction term, and thus should be called interaction quantum potential.

The imaginary part of the Schrödinger equation yields

$$\frac{\partial R^2}{\partial t} + \nabla \cdot \left(R^2 \frac{\nabla S}{m} \right) + \frac{2R}{\hbar} \, \mathrm{Im} \left(e^{-\frac{i}{\hbar} S} V \star \left(R e^{\frac{i}{\hbar} S} \right) \right) = 0, \tag{4.12}$$

which is identical to equation (4.6) for probability conservation, now written in function of the R- and S-fields.

Returning to the equation (4.8), we see that while V_{nc} and Q_K have their partners in NCQFT, Q_I, on the other hand, has no analog when a comparison is done with field theory. It could be written as $Q_I(x^i, S, \partial^i S, \partial^{ij} S, \partial^{ijk} S, \ldots, R, \partial^i R, \partial^{ij} R, \partial^{ijk} R, \ldots)$, keeping in mind, of course, that the higher derivative interactions come from a star product of

V with the Ψ field, and therefore Q_I is not exactly an arbitrary function of x^i, S, R and their partial derivatives. When V is polynomial, Q_I contains a finite number of derivatives, otherwise it will be infinite series containing derivatives of all orders, which characterizes a interaction nonlocal in space. This case can still be analyzed, at least perturbatively. A general discussion of how to handle mathematically nonlocal particle mechanics and field theory may be found in [39, 40], and in the context of noncommutative theories in [41]. The reason for the absence of the interaction quantum potential Q_I in the case of NCQFT is that in that case there is no star product between the potential and the wavefunctional in the Schrödinger equation. In the case of quantum field theory, the potential term is valued on the star algebra and its product with the wavefunctional is the ordinary one, while, in the quantum mechanical case, the potential is valued on the ordinary algebra and multiplies the wave function with the star product.

Notice that, in case of considering quantum field theory with an external potential, it will be incorporated in the quadratic term of the hamiltonian with the star product, but its product with the wavefunctional will still be the ordinary one. This point sets a difference between the implications of noncommutativity for high and low energy phenomena. At least in what concerns the structure of the equations in the Hamilton-Jacobi formalism, the complexity and richness of the new physics originated from the noncommutativity may be manifest in a more impressive way in the low energy processes.

4.3 The classical and commutative limits

The most quoted criterion for finding the classical limit of a quantum-mechanical systems is $\hbar \to 0$. As pointed out when discussing the NCQFT, the adoption of this criterion as universal for the classical limit has several problems. For the determination of the commutative limit, the most employed criterion is $\theta^{ij} \to 0$ [1, 2]. However, arguments for its failure in the most general case were presented in the framework of NCQFT in the Third section. The arguments presented there are still valid and reinforced here, where they become yet stronger by the presence of the additional quantum potential Q_I and the *equivariance* correction Σ_θ. Among other criteria commonly used for the classical limit are high quantum numbers ($n \to \infty$), large mass ($m \to \infty$), and short de Broglie wavelength. All of these criteria are not universal, and can be obtained from the application for particular systems of the most general and systematic method based on the generalized Hamilton-Jacobi formalism. The latter, according to the case under consideration, provides the best choice as the parameter to vary.

The conditions for the attainment of the classical and commutative limits of a quantum-mechanical system can be obtained easily from the examination of the system of equations constituted by (4.8) and (4.12). For simplicity, we shall restrict our considerations to the case where $\Sigma_\theta \to 0$, which will be the one of interest for the examples that follow. According to the prescription for the identification of the quantum effects proposed in section 3, and applied in this subsection to NCQM, all the quantum manifestations of a given noncommutative quantum-mechanical system must be governed by Q_K and Q_I Thus, in the limit where $Q_K + Q_I \to 0$ the system is expected to present a classical behavior.

By looking for the criteria for the commutative limit, however, we see that the conditions that must be imposed on the set $\{V_{nc}, Q_K, Q_I\}$ are less trivial. While useful for the formal analysis comparing NCQM with NCQFT, the grouping of the quantum contributions in the two potentials Q_K and Q_I is not convenient for the identification of the noncommutative quantum effects. This can be done by following an approach similar to the one adopted for NCQFT. We define the noncommutative quantum contributions by

$$Q_{nc} = Q_K + Q_I - Q_c \,, \tag{4.13}$$

where

$$Q_c = -\frac{\hbar^2}{2m}\frac{\nabla^2 R_c}{R_c}\,, \qquad R_c = \sqrt{\Psi_c^* \Psi_c}\,.$$

Ψ_c is the wavefunction obtained from the commutative Schrödinger equation containing the usual potential $V(x^i)$, that is, the equation obtained by making $\theta^{ij} = 0$ on (4.1) before solving it. The conditions to assure that (4.8) and (4.12) assume a form identical to the one corresponding to a commutative system are

$$V_{nc} + Q_{nc} \to 0\,. \tag{4.14}$$

The condition for the classical limit can now be written as $Q_c + Q_{nc} \to 0$, and (4.8) as

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V + V_{nc} + Q_c + Q_{nc} = 0\,. \tag{4.15}$$

5. Simple applications

5.1 Noncommutative harmonic oscillator

Here we show a simple application of the ideas presented in the latter section. Consider a two dimensional noncommutative harmonic oscillator. In two dimensions, (1.1) can be written as

$$\left[\widehat{X}^\mu, \widehat{X}^\nu\right] = i\theta\epsilon^{\mu\nu}\,. \tag{5.1}$$

The hamiltonian of the system is given by

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}mw^2\left[\left(x - \frac{\theta}{2\hbar}p_y\right)^2 + \left(y + \frac{\theta}{2\hbar}p_x\right)^2\right], \tag{5.2}$$

where m is the particle mass and w the frequency of the associated commutative oscillator.

The corresponding Schrödinger equation in polar coordinates is

$$\begin{aligned} i\hbar\frac{\partial\Psi_\theta(r,\varphi,t)}{\partial t} &= H_\theta\Psi_\theta(r,\varphi,t) \\ &= -\frac{\hbar^2}{2m}\left(1 + \left(\frac{mw\theta}{2\hbar}\right)^2\right)\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\varphi^2\right)\Psi_\theta(r,\varphi,t) + \\ &\quad + \left(i\frac{m}{2}\theta w^2\partial_\varphi + \frac{m}{2}w^2 r^2\right)\Psi_\theta(r,\varphi,t)\,. \end{aligned} \tag{5.3}$$

By plugging $\Psi_\theta(r,\varphi,t) = e^{-iEt/\hbar}\psi_\theta(r,\varphi)$ into (5.3) we obtain the eigenvalue equation $H_\theta\psi_\theta(r,\varphi) = E_\theta\psi_\theta(r,\varphi)$. The solution of this eigenvalue equation is straightforward (see, for example [42]), and gives

$$\psi_\theta(r,\varphi) = \psi_{n,\alpha,\theta}(r,\varphi) = (-1)^n\sqrt{\frac{n!\tilde{\zeta}}{\pi(n+|\alpha|)!}}\exp\left(-\frac{\tilde{\zeta}r^2}{2}\right)\left(\sqrt{\tilde{\zeta}}r\right)^{|\alpha|}L_{n,\theta}^{|\alpha|}\left(\tilde{\zeta}r^2\right)e^{i\alpha\varphi}, \tag{5.4}$$

where $L_{n,\theta}^{|\alpha|}\left(\tilde{\zeta}r^2\right)$ are the associated Laguerre polynomials

$$L_{n,\theta}^{|\alpha|}\left(\tilde{\zeta}r^2\right) = \sum_{l=0}^{n}(-1)^l\begin{pmatrix} n+|\alpha| \\ n-l\end{pmatrix}\frac{\left(\tilde{\zeta}r^2\right)^l}{l!}, \qquad \tilde{\zeta}^2 = \frac{\left(\frac{mw}{\hbar}\right)^2}{1+\left(\frac{mw\theta}{2\hbar}\right)^2}, \tag{5.5}$$

$n = 0,1,2,\ldots$ is the principal quantum number, and $\alpha = 0,\pm1,\pm2\ldots$is the angular canonical momentum quantum number. The corresponding energy spectrum is

$$E_{n,\alpha,\theta} = 2\hbar w\left(1-\left(\frac{mw\theta}{2\hbar}\right)^2\right)^{1/2}\left(n+\frac{|\alpha|+1}{2}\right)-\frac{m\theta w^2\alpha}{2}. \tag{5.6}$$

Let us consider the state where $n = 0$, whose wavefunction is

$$\Psi_\theta(r,\varphi,t) = \sqrt{\frac{\tilde{\zeta}}{\pi|\alpha|!}}\exp\left(-\frac{\tilde{\zeta}r^2}{2}\right)\left(\sqrt{\tilde{\zeta}}r\right)^{|\alpha|}e^{i\alpha\varphi-iEt/\hbar}. \tag{5.7}$$

For this physical state, the corresponding V , V_{nc}, Q_c, Q_{nc} and Σ_θ are

$$\begin{aligned}
V &= \frac{1}{2}mw^2r^2 \\
V_{nc} &= \left(\frac{mw\theta}{2\hbar}\right)^2\frac{\alpha^2\hbar^2}{2mr^2}-\frac{m\theta w^2\alpha}{2} \\
Q_c &= -\frac{1}{2}mw^2r^2+\hbar w(|\alpha|+1)-\frac{\alpha^2\hbar^2}{2mr^2} \\
Q_{nc} &= \left[\sqrt{1-\left(\frac{mw\theta}{2\hbar}\right)^2}-1\right]\hbar w(|\alpha|+1)-\left(\frac{mw\theta}{2\hbar}\right)^2\frac{\alpha^2\hbar^2}{2mr^2} \\
\Sigma_\theta &= 0.
\end{aligned} \tag{5.8}$$

A first inspection on the potentials shows that in the lowest energy state, characterized by $\alpha = 0$ and $r = 0$, the zero point energy is given by

$$E_{0,0,\theta} = Q_c + Q_{nc} = \hbar w\left(1-\left(\frac{mw\theta}{2\hbar}\right)^2\right)^{1/2}, \tag{5.9}$$

exactly what would be expected from quantum potentials, which must account for all nonclassical behaviors.

According to the criteria proposed in the latter subsection, the condition for the classical limit is $Q_c + Q_{nc} \to 0$, while for the commutative limit the condition is reduced to $V_{nc} + Q_{nc} \to 0$. We start by the commutative limit. The sum $V_{nc} + Q_{nc}$ can be written as

$$V_{nc} + Q_{nc} = -\left(\frac{mw\theta}{2\hbar}\right)\hbar w\alpha + \left[\sqrt{1-\left(\frac{mw\theta}{2\hbar}\right)^2} - 1\right]\hbar w\left(|\alpha|+1\right). \tag{5.10}$$

The sum of other terms in the hamiltonian of (4.15) is

$$\frac{(\nabla S)^2}{2m} + V + Q_c = \hbar w\left(|\alpha|+1\right) \tag{5.11}$$

In order that (5.10) be negligible when compared to (5.11), it is sufficient that the parameters $m, w, \hbar$ and θ satisfy $mw\theta/2\hbar \ll 1$, or in other words, that $\theta \ll 2\hbar/mw$. This is the precise meaning that must be attributed to $\theta \to 0$ in the case of the noncommutative harmonic oscillator. Once this condition is satisfied, all the observable quantities computed with the aid of the oscillator wavefunction, as expected values and also the spectrum, will be indistinguishable to the ones corresponding to its commutative counterpart.

There is a general comment that, what is important to recover the classical world, is the smallness of $\hbar$ with respect to the physical quantities of the problem.[5] As stated in when performing a theoretical analysis in section 3, and reinforced in subsection 4.3, that statement is ungrounded. By looking for the classical limit of the noncommutative harmonic oscillator, it is easy to see that there is no parameter to vary in order that the system assumes a classical behavior without canceling the interaction term, which would be a trivial and uninteresting choice. Since Q_c+Q_{nc} contains a term, $-mw^2r^2/2$, which do not depend on $\hbar$, no matter how small, but non-null, this constant is considered, the quantum effects cannot be neglected. The classical limit of the harmonic oscillator in the context of ordinary commutative quantum mechanics is only achieved by constructing coherent states, which are superpositions of states containing different quantum numbers [43]. For the case of NCQM, however, that procedure is non-trivial, and involves certain subtleties [44].

This example of application is useful to illustrate the spirit of the ideas defended in this work, that the commutative and classical limits of a system must be considered as realized on its physical states, rather on the equations of motion.

5.2 WKB approximation in the case of a one-direction interacting potential

Here, we present an example to complete the one of the latter subsection in a discussion about noncommutative/commutative and quantum/classical passages. We shall show how the commutative limit may be obtained in certain cases by varying parameters other than θ.

Consider a particle in two dimensional plane under the action of a one-direction interacting potential. The noncommutative Schrödinger equation is given by

$$i\hbar\frac{\partial}{\partial t}\Psi = \left[\frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + V\left(x - \frac{\theta\hat{p}_y}{2\hbar}\right)\right]\Psi\,. \tag{5.12}$$

[5] In the notation adopted in this work, such a claim is written as $\hbar \to 0$.

Writing the wavefunction as

$$\Psi\left(x,y,t\right)=e^{-iEt/\hbar+ik_y y}\psi(x)\,, \tag{5.13}$$

we separate the variables, obtaining

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V\left(x-\frac{\theta k_y}{2}\right)\psi\;=\;E_x\psi \tag{5.14}$$

$$\frac{\hbar k_y}{2m}\;=\;E_y\,. \tag{5.15}$$

Performing a WKB approximation inserting the *ansatz*

$$\psi(x)=e^{i\phi(x)/\hbar}\,,\qquad \phi=\phi_0+\hbar\phi_1+\cdots \tag{5.16}$$

on (5.14) we find, after simplification

$$\Psi^{\pm}_{WKB}(x,y,t)=\frac{C_{\pm}\exp\left[\frac{i}{\hbar}\left(\pm\int 2m\left(E_x-V-V_{nc}\right)^{1/2}dx-\left(E_x+E_y\right)t+\hbar k_y y\right)\right]}{\left[2m\left(E_x-V-V_{nc}\right)\right]^{1/4}}\,, \tag{5.17}$$

where $C_{\pm}$ is a constant.

The conditions for the validity of this approximation are (see, for example, [26])

$$\begin{aligned}
\frac{m\hbar\left|\left(V+V_{nc}\right)'\right|}{\left[2m\left(E_x-V-V_{nc}\right)\right]^{3/2}}&\ll 1\,,\\
\frac{m\hbar^2\left|\left(V+V_{nc}\right)''\right|}{\left[2m\left(E_x-V-V_{nc}\right)\right]^{2}}&\ll 1\,,\\
\frac{m\hbar^3\left|\left(V+V_{nc}\right)'''\right|}{\left[2m\left(E_x-V-V_{nc}\right)\right]^{5/2}}&\ll 1\,,
\end{aligned} \tag{5.18}$$

and so on for higher derivatives of $\left(V+V_n\right)'=d\left(V+V_{nc}\right)/dx$. The potentials corresponding to the wavefunction (5.17) are

$$\begin{aligned}
V_{nc}&=V\left(x-\frac{\theta k_y}{2}\right)-V(x)\\
Q_c&=-\frac{\hbar^2}{2m}\left(E_x-V\right)^{1/4}\frac{d^2}{dx^2}\left(E_x-V\right)^{-1/4}\\
Q_{nc}&=-\frac{\hbar^2}{2m}\left(E_x-V-V_{nc}\right)^{1/4}\frac{d^2}{dx^2}\left(E_x-V-V_{nc}\right)^{-1/4}+\\
&\quad+\frac{\hbar^2}{2m}\left(E_x-V\right)^{1/4}\frac{d^2}{dx^2}\left(E_x-V\right)^{-1/4}\\
\Sigma_\theta&=0\,.
\end{aligned} \tag{5.19}$$

Notice that θ appears multiplying $k_y=k\sin\varphi$, where φ is the angle between the direction of motion and the direction where the potential acts. It can thus be chosen as our parameter to vary in order to measure the degree of deviation from the commutative behavior. Let us suppose that θ is sufficiently small (as it should be) in order that (4.5) be undetectable, but "big" enough to have some influence on other detectable quantities, such as energy spectrum, etc. If φ is small and renders $V_{nc}+Q_{nc}$ negligible as compared to the other terms of the hamiltonian in the Hamilton-Jacobi equation, the system assumes a commutative behavior. When the speed of the particle is sufficiently slow, this limit is achieved for relatively large angles.

From the conditions (5.18), one can infer that $|Q_c+Q_{nc}| \ll E_x - V - V_{nc}$, and thus the wavefunction is semiclassical if they are satisfied. Observe that the turning points, where $E_x - V - V_{nc} = 0$ and the approximation fails, are shifted in comparison with the ones of the analogous commutative problem due to the presence of V_{nc}.

6. Discussion and outlook

In this work, we presented an alternative interpretation for the meaning of the canonical noncommutativity obtained from arguments of string theory. In order to justify our point of view on the validity of interpreting (1.1) as being realized just by the coordinate observables of the particles, and not by the spacetime itself, we discussed some of the aspects of the derivation of the noncommutativity in the Landau problem and in the stringy context. Arguments for the existence of a loophole in the usual interpretation of the results found in the literature were presented.

When we consider the stringy noncommutativity from the new point of view, the interpretation of the Weyl symbols as points of the spacetime, considered as senseless in [13, 33, 36], but adopted by the great majority of the string inspired articles, like [8], for example, is perfectly justified. Although these articles do not presuppose our interpretation for noncommutativity, their calculations are in accordance with it. The possible problem in the legitimacy of the use of the Feynman graph approach on noncommutative spaces, which was underlined in a discussion carried on in [15], is absent here, since NCQFT is assumed as being based on a commutative spacetime.

In the context of quantum mechanics, the implications of adopting the new point of view for noncommutativity are manifest into the foundations. The common assertion that the concept wavefunction does not make sense in noncommutative spaces [13, 36] looses its validity for the stringy noncommutativity considered because it is not of the spacetime coordinates. The wavefunction, however, has a different meaning in NCQM than in its commutative counterpart. It is possible to define a consistent probability density given by the expression $\rho = |\Psi|^2$, but the interpretation of $\rho(\vec{x},t)d^3x$ is that of being the probability of finding the canonical coordinate of the particle in the volume d^3x around the point $\vec{x}$ at time t.

Other related topic of interest discussed was the use of the Schrödinger formulation and the associated Hamilton-Jacobi formalism to perform an analysis of the structure of NCQFT and NCQM and their possible limiting behaviors. A formal analysis, comparing the potential terms according to the derivatives present in their definitions, revealed that NCQM is richer than NCQFT from this point of view. While the complete description of NCQFT can be done by considering the properties of the three potentials V, V_{nc} and Q_K, the corresponding one for NCQM is more complex, needing the four potentials V, V_{nc}, Q_K, Q_I and the Σ_θ term present in the probability conservation equation.

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The conditions for the attainment of the classical and commutative limits of NCQFT were determined as being $Q_K = Q_c + Q_{nc} \to 0$ and $V_{nc} + Q_{nc} \to 0$, respectively. For NCQM, our considerations were restricted to the case where $\Sigma_\theta \to 0$, which was the one of interest for the examples of application presented. The conditions for the classical limit were found to be $Q_K + Q_I = Q_c + Q_{nc} \to 0$, and for the commutative one $V_{nc} + Q_{nc} \to 0$.

In the same way that $\hbar$ must not be considered as the agent responsible for the quantum effects in ordinary and noncommutative field theories and quantum mechanics, θ^{ij} must not be considered as the one responsible for the noncommutative manifestations in NCQFT and NCQM. Although achieving the commutative limit by setting $\theta^{ij} \to 0$ was possible in the examples presented, this route for the classical limit will not always be available when hamiltonians containing more complicated potentials are considered [45]. In these cases the formalism of this work may be useful in finding other routes for the attainment of the commutative behavior.

There is the immediate possibility to apply the Hamilton-Jacobi formalism present in this work to investigate the properties of some simple models of noncommutative scalar field theory or noncommutative quantum mechanical systems whose hamiltonians possesses higher-order derivatives or are nonlocal, like, for example, the anharmonic oscillator and the hydrogen atom. The extension of the ideas presented here is in principle possible for other theories than the scalar field ones, like the noncommutative version of QED.

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